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Abstract: The present research grew out of the authors' joint work [M. Brodmann, P. Schenzel, Arithmetic properties of projective varieties of almost minimal degree, *J. Algebraic Geom.*, in press]. It continues the study of the structure of projective varieties of almost minimal degree, focusing to the case of small codimension. In particular, we give a complete list of all occurring Betti diagrams in the cases where $\text{codim } X \leq 4$.

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ON VARIETIES OF ALMOST MINIMAL DEGREE IN SMALL CODIMENSION

MARKUS BRODMANN, PETER SCHENZEL

1. INTRODUCTION

Let $X \subset \mathbb{P}_K^r$ denote an irreducible and reduced projective variety over an algebraically closed field K . In the following we always assume that X is non-degenerate, that is not contained in a hyperplane. Then, the degree and the codimension of X satisfy the inequality $\deg X \geq \operatorname{codim} X + 1$ (cf. for instance [9]). Varieties for which equality holds are called varieties of *minimal degree*. They are completely classified (cf. for instance [7] and [9]). In particular they are arithmetically Cohen-Macaulay and have a linear minimal free resolution. That is, their Betti numbers are explicitly known.

In case $\deg X = \operatorname{codim} X + 2$, the variety X is called a variety of *almost minimal degree*. Here one has a much greater variety for the Betti numbers. In the framework of polarized varieties of Δ -genus 1, Fujita (cf. [3] and [4]) provides a satisfactory description of varieties of almost minimal degree. The study of varieties of almost minimal degree is pursued by the authors (cf. [2]) from the arithmetic point of view. It turns out (cf. [2]) that $X \subset \mathbb{P}_K^r$ is either an arithmetically normal Del Pezzo variety or a proper projection of a variety of minimal degree. See also 3.1 for the precise statement.

The investigation of homological properties of varieties of almost minimal degree was initiated by Hoa, Stückrad, and Vogel (cf. [10]). We refer also to [2] and [12] for certain improvements of their results about the Betti numbers. In particular the Castelnuovo-Mumford regularity of a variety X of almost minimal degree satisfies $\operatorname{reg} X \leq 2$ (cf. [6] for the definition of the Castelnuovo-Mumford regularity).

The aim of the present exposition is to investigate varieties of almost minimal degree and of low codimension, in particular their Betti diagrams. More precisely, we describe the structure of the minimal free resolution of a variety X of almost minimal degree of $\operatorname{codim} X \leq 4$ by listing the possible Betti diagrams. Let us recall that the structure of arithmetically Cohen-Macaulay resp. Gorenstein varieties in codimension 2 resp. 3 is known by the Theorems of Hilbert-Burch resp. Buchsbaum-Eisenbud (cf. [6]). So, we need not to discuss these cases in detail any more. As the Betti diagrams, the degree and the codimension are not affected if X is replaced by a cone over X , we shall assume that X is not a cone. The most surprising fact is, that the non-arithmetically Cohen-Macaulay case can occur only in small dimensions (cf. Section 2 for the precise statements). Our main technical tool is a result shown by the authors (cf. [2]), which says that besides of an exceptional case, (that is the generic projection of the Veronese surface in \mathbb{P}_K^5) any non-arithmetically normal (and in particular non-arithmetically Cohen-Macaulay) variety of

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almost minimal degree $X \subset \mathbb{P}_K^r$ (which is not a cone) is contained in a variety of minimal degree $Y \subset \mathbb{P}_K^r$ such that $\text{codim}(X, Y) = 1$.

The present research grew out of the authors' joint work [2]. It continues with the study of the structure of varieties of almost minimal degree, in particular in small codimension. The authors thank Uwe Nagel for making his preprint [12] available.

2. MAIN RESULTS

Let $X \subset \mathbb{P}_K^r$ denote a non-degenerate variety of almost minimal degree, hence an integral closed subscheme with $\deg X = \text{codim } X + 2$ not contained in a hyperplane $\mathbb{P}_K^{r-1} \subset \mathbb{P}_K^r$. We use the abbreviations $\dim X = d$ and $\text{codim } X = c$. Let $S = K[x_0, \dots, x_r]$ denote the polynomial ring in $r + 1$ variables, so that $\mathbb{P}_K^r = \text{Proj}(S)$. Let $A_X = S/I_X$ denote the homogeneous coordinate ring of X , where $I_X \subset S$ is the defining ideal of X . The codepth A_X of A_X is defined as the difference $\text{codepth } A_X := \dim A_X - \text{depth } A_X$, where $\text{depth } A_X$ denotes the depth of A_X .

For the notion of Betti diagrams we follow the suggestion of Eisenbud (cf. [6]). That is, in a diagram the number in the i -th column and the j -th row is

$$\dim_K \text{Tor}_i^S(K, I_X)_{i+j}.$$

Outside of the diagram's range all the entries are understood to be zero.

Theorem 2.1. ($\text{codim } X = 2$) *Let $X \subset \mathbb{P}_K^r$ be a non-degenerate variety of degree 4 and codimension 2 which is not a cone. Then X is of one of the following types:*

- (a) *X is a complete intersection cut out by two quadrics.*
- (b) *$\dim X \leq 4$ and the Betti diagram of I_X has the form*

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 0 \\ 2 & 3 & 4 & 1 \end{array}.$$

- (c) *(The exceptional case) X is a generic projection of the Veronese surface $F \subset \mathbb{P}_K^5$ and the Betti diagram of I_X has the shape*

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 7 & 10 & 5 & 1 \end{array}.$$

Moreover for any $1 \leq d \leq 4$ there are examples as mentioned in (b) such that $\dim X = d$.

In the case where X is a Cohen-Macaulay variety, Theorem 2.1 (b) has been shown by Nagel (cf. [12]). Under this additional assumption one has $\dim X \leq 2$. Theorem 2.1 grew out of our understanding of Nagel's arguments. Our approach enables us to investigate the cases of codimension three and four as well.

Theorem 2.2. ($\text{codim } X = 3$) *Let $X \subset \mathbb{P}_K^r$ denote a non-degenerate variety of degree 5 and codimension 3 which is not a cone. Then X is of one of the following types:*

- (a) *X is the Pfaffian variety defined by the five Pfaffians of a skew symmetric 5×5 matrix. Its minimal free resolution is determined by the Buchsbaum-Eisenbud complex.*

- (b) $\dim X \leq 4$ and $\text{codepth } A_X = 1$. The Betti diagram of the defining ideal I_X has the following form

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 4 | 2 | 0 | 0 |
| 2 | 1 | 6 | 5 | 1 |

- (c) $\dim X \leq 5$ and $\text{codepth } A_X = 2$. The Betti diagram of I_X has the following shape

| | 1 | 2 | 3 | 4 | 5 |
|---|---|----|----|---|---|
| 1 | 3 | 2 | 0 | 0 | 0 |
| 2 | 6 | 16 | 15 | 6 | 1 |

For any of the dimensions and codepths admitted in (b) and (c) resp. there are examples of varieties of almost minimal degree.

The next result concerns the case of codimension 4. A new phenomenon occurs in this situation. Namely there are two different shapes of Betti diagrams for the same codepth.

Theorem 2.3. ($\text{codim } X = 4$) Let $X \subset \mathbb{P}_K^r$ denote a non-degenerate variety of degree 6 and codimension 4 which is not a cone. Then X is one of the varieties of the following list:

- (a) X is arithmetically Gorenstein. Its minimal free resolution has the following form

$$0 \rightarrow S(-6) \rightarrow S^9(-4) \rightarrow S^{16}(-3) \rightarrow S^9(-2) \rightarrow I_X \rightarrow 0.$$

- (b) $\dim X \leq 4$ and $\text{codepth } A_X = 1$. The Betti diagram of the defining ideal I_X has one of the following two forms:

| | 1 | 2 | 3 | 4 | 5 |
|---|---|----|----|---|---|
| 1 | 8 | 12 | 3 | 0 | 0 |
| 2 | 1 | 4 | 10 | 6 | 1 |

resp.

| | 1 | 2 | 3 | 4 | 5 |
|---|---|----|----|---|---|
| 1 | 8 | 11 | 3 | 0 | 0 |
| 2 | 0 | 4 | 10 | 6 | 1 |

- (c) $\dim X \leq 5$ and $\text{codepth } A_X = 2$. The Betti diagram of I_X is of the form:

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|----|----|----|---|---|
| 1 | 7 | 8 | 3 | 0 | 0 | 0 |
| 2 | 3 | 19 | 30 | 21 | 7 | 1 |

- (d) $\dim X \leq 6$ and $\text{codepth } A_X = 3$. The Betti diagram of I_X has the following shape:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|----|----|----|----|----|---|---|
| 1 | 6 | 8 | 3 | 0 | 0 | 0 | 0 |
| 2 | 10 | 40 | 65 | 56 | 28 | 8 | 1 |

For any of the dimensions and codepths admitted in (b), (c) and (d) resp. there are examples of varieties of almost minimal degree.

The final result of this note concerns varieties of almost minimal degree and of codimension less than half the embedding dimension. First of all note (cf. Example 4.8) that in codimension 6 there are examples of a fixed codepth with rather different Betti diagrams. That is, the corresponding statements to Theorem 2.1, 2.2 and 2.3 are not true in larger codimension.

Corollary 2.4. *Let $X \subset \mathbb{P}_K^r$ be a non-degenerate variety of almost minimal degree which is not a cone such that $\text{codim } X \geq 3$. Suppose that $\dim X > \text{codim } X + 2$. Then $\text{codim } X = 3$ and X is arithmetically Gorenstein. Therefore X is defined by the five Pfaffians of a skew symmetric 5×5 matrix.*

Moreover, in the case where X is a variety of almost minimal degree and $\dim X \leq \text{codim } X + 2$, that is when X is not necessary a Del Pezzo variety there are estimates about the Betti numbers and their vanishing (cf. 3.3 for the details).

3. OUTLINE OF THE PROOFS

Let $X \subset \mathbb{P}_K^r$ denote a non-degenerate reduced irreducible variety of almost minimal degree. By the work of the authors (cf. [2]) it follows that X is either a normal Del Pezzo variety – in this case X is arithmetically Gorenstein – or a projection of a variety of minimal degree.

First, we consider the case of non-arithmetically Cohen-Macaulay varieties of almost minimal degree. In this situation we have the following characterization in which $\text{Sec}_P(Z)$ denotes the secant cone of a projective variety Z with respect to a point P in the ambient space.

Lemma 3.1. *Let $X \subset \mathbb{P}_K^r$ denote a non-degenerate reduced irreducible variety. Let $1 \leq t \leq \dim X =: d$ denote an integer. Then the following conditions are equivalent:*

- (i) $\deg X = \text{codim } X + 2$ and $\text{depth } A_X = t$.
- (ii) X is the projection of a variety $Z \subset \mathbb{P}_K^{r+1}$ of minimal degree from a point $P \in \mathbb{P}_K^{r+1} \setminus Z$ such that $\dim \text{Sec}_P(Z) = t - 1$.

Moreover, $1 \leq \text{depth } A_X \leq 4$.

Proof. Cf. [2, Theorem 1.1, Theorem 1.2 and Corollary 7.5]. □

In view of Lemma 3.1 there is some need for information about varieties of minimal degree in order to understand varieties of almost minimal degree. A variety of minimal degree $Z \subset \mathbb{P}_K^s$ is either

- a quadric hypersurface,
- a (cone over a) Veronese surface in \mathbb{P}_K^5 , or
- a (cone over a) smooth rational normal scroll

(cf. [7] and [9] for the details and the history of this classification).

Next we recall a few basic facts about rational normal scrolls (cf. also [9]). A (cone over a) rational normal scroll $S(a_1, \dots, a_k)$ is defined as the "rank two subscheme in \mathbb{P}_K^s " defined by the matrix

$$\begin{pmatrix} x_{10} & \dots & x_{1a_1-1} & \vdots & x_{20} & \dots & x_{2a_2-1} & \vdots & \dots & \vdots & x_{k0} & \dots & x_{ka_k-1} \\ x_{11} & \dots & x_{1a_1} & \vdots & x_{21} & \dots & x_{2a_2} & \vdots & \dots & \vdots & x_{k1} & \dots & x_{ka_k} \end{pmatrix}$$

with $s = k - 1 + \sum_{i=1}^k a_i$. It is well known that $\dim S(a_1, \dots, a_k) = k$ and therefore

$$\deg S(a_1, \dots, a_k) = \text{codim } S(a_1, \dots, a_k) + 1 = \sum_{i=1}^k a_i,$$

so that $S(a_1, \dots, a_k)$ is a variety of minimal degree. Keep in mind that $S(a_1, \dots, a_k)$ is a proper cone if and only if $a_i = 0$ for some $i \in \{1, \dots, k\}$.

Moreover we need some information about the Hilbert series $F(\lambda, A_X)$ of the graded K -algebra A_X . Let us recall that the Hilbert series of a graded K -algebra A is the formal power series defined by

$$F(\lambda, A) = \sum_{i \geq 0} (\dim_K A_i) \lambda^i.$$

The Hilbert series of a variety of almost minimal degree may be described as follows.

Lemma 3.2. *Let $X \subset \mathbb{P}_K^r$ denote a variety of almost minimal degree. Put $q = \text{codepth } A_X$. Then*

$$F(\lambda, A_X) = \frac{1}{(1 - \lambda)^{d+1}} (1 + (c + 1)\lambda - \lambda(1 - \lambda)^{q+1}),$$

where $c = \text{codim } X$ and $d = \dim X$. Furthermore $\dim_K (I_X)_2 = \binom{c+1}{2} - q - 1$.

Proof. Cf. [2, Corollary 4.4]. □

As a consequence of Lemma 3.1 the authors (cf. [2]) derived some information about the Betti numbers of I_X for certain varieties $X \subset \mathbb{P}_K^r$ of almost minimal degree.

Lemma 3.3. *Let $X \subset \mathbb{P}_K^r$ be a variety of almost minimal degree which is not arithmetically Cohen-Macaulay. Suppose that X is not a generic projection of (a cone over) the Veronese surface $F \subset \mathbb{P}_K^5$. Then there exists a variety of minimal degree $Y \subset \mathbb{P}_K^r$ such that $X \subset Y$ and $\text{codim}(X, Y) = 1$. Moreover*

$$\text{Tor}_i^S(k, A_X) \simeq k^{u_i}(-i-1) \oplus k^{v_i}(-i-2) \text{ for } 1 \leq i \leq c+q,$$

where $c = \text{codim } X$, $q = \text{codepth } A_X$ and

- (a) $u_1 = \binom{c+1}{2} - q - 1$,
 $i \binom{c}{i+1} \leq u_i \leq (c+1) \binom{c}{i} - \binom{c}{i+1}$, if $1 < i < c - q$,
 $u_i = i \binom{c}{i+1}$, if $c - q \leq i < c$,
 $u_i = 0$, if $c \leq i \leq c + q$.
- (b) $\max\{0, \binom{c+q-1}{i+1} - (i+2) \binom{c}{i+1}\} \leq v_i \leq \binom{c+q+1}{i+1}$, if $1 \leq i < c - q - 1$,
 $v_i = \binom{c+q+1}{i+1} - (i+2) \binom{c}{i+1}$, if $\max\{1, c - q - 1\} \leq i < c$,
 $v_i = \binom{c+q+1}{i+1}$, if $c \leq i \leq c + q$.

In addition $v_i - u_{i+1} = \binom{c+q+1}{i+1} - (c+1) \binom{c}{i+1} + \binom{c}{i+2}$ for all $1 \leq i < c$.

Proof. Cf. [2, Theorem 1.1 and Theorem 8.3]. □

In the particular case where $\dim X = 1$ the statement about the Betti numbers has been shown independently by Nagel (cf. [12]).

In the following remark, we add a comment concerning the exceptional case of a generic projection of (a cone over) the Veronese surface $F \subset \mathbb{P}_K^5$ and the normal case.

Remark 3.4. A) (The exceptional case) Let $F \subset \mathbb{P}_K^5$ be the Veronese surface defined by the 2×2 -minors of the symmetric matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix}.$$

Let $P \in \mathbb{P}_K^5 \setminus F$ denote a point. Suppose that $\text{rank } M|_P = 3$, that is the case of a generic point. Remember that $\det M = 0$ defines the secant variety of F . Then the projection of F from P defines a surface $X \subset \mathbb{P}_K^4$ of almost minimal degree and $\text{depth } A_X = 1$. The surface X is cut out by seven cubics (cf. 3.2), so that it is not contained in a variety of minimal degree.

B) (The arithmetically Cohen-Macaulay case) Let $X \subset \mathbb{P}_K^r$ denote an arithmetically normal variety of almost minimal degree. Then, (cf. [2]) X is a Del Pezzo variety and hence arithmetically Gorenstein. In general X is not a one codimensional subvariety of a variety of minimal degree. To this end let $X \subset \mathbb{P}_K^9$ be the smooth codimension three variety cut out by the 4×4 -Pfaffians of a generic skew-symmetric 5×5 -matrix. Then X is not contained in a variety Y of minimal degree such that $\text{codim}(X, Y) = 1$ (cf. [2] for the details).

Proofs. After these preparations, we now come to the proofs of our Theorems. In fact, the proofs of statements 2.1 (a), 2.2 (a), and 2.3 (a) are easy by well-known results. In fact the structure of the minimal free resolution of I_X given in Theorem 2.3 (a) is a consequence of [13, Theorem B].

Next we prove (c) of Theorem 2.1. To this end let $X \subset \mathbb{P}_K^4$ be a generic projection of the Veronese surface $F \subset \mathbb{P}_K^5$. Then $\dim X = 2$, $\text{codim } X = 2$ and $\text{depth } A_X = 1$. As seen above, I_X does not contain any quadric. Therefore I_X has a linear resolution. Remember that $\text{reg } I_X = 3$ (cf. [2]). A computation with the aid of the Hilbert series (cf. 3.2) proves the structure of the Betti diagram of 2.1 (c).

For all other statements of Theorems 2.1, 2.2 and 2.3 we may assume that the variety of almost minimal degree $X \subset \mathbb{P}_K^r$ is not arithmetically Cohen-Macaulay and not a projection of the Veronese surface $F \subset \mathbb{P}_K^5$. So, X is contained in a variety of minimal degree Y of codimension $c - 1$ (cf. 3.3). That is, Y is defined as the zero locus of $\binom{c}{2}$ quadrics. On the other hand the defining ideal I_X is generated by $\binom{c+1}{2} - q - 1$ quadrics (cf. 3.2). This implies that

$$\dim_K(I_X/I_Y)_2 = c - q - 1 \geq 0,$$

where $c = \text{codim } X$, $q = \text{codepth } A_X \geq 1$. Considering all possibilities that arise for $c = 2, 3, 4$ it follows that, with the exception of $c = 4, q = 1$, the general result (cf. 3.3) determines the corresponding Betti diagrams.

The particular case of $c = 4, q = 1$, yields the following shape of the Betti diagram

| | 1 | 2 | 3 | 4 | 5 |
|---|-------|-------|----|---|---|
| 1 | 8 | u_2 | 3 | 0 | 0 |
| 2 | v_1 | 4 | 10 | 6 | 1 |

with $v_1 - u_2 = -11$. In order to finish the proof we observe that $v_1 \leq 1$ (cf. Lemma 3.5).

To complete the proof of Theorems 2.1, 2.2 and 2.3 we have to prove the stated constraints on the occurring dimensions and codepths. To do so, we may assume that X is not

arithmetically Cohen-Macaulay. Therefore (cf. Lemma 3.1) X is the projection of a variety of minimal degree $Z \subset \mathbb{P}_K^{r+1}$ from a point $P \in \mathbb{P}_K^{r+1} \setminus Z$ such that $\dim \text{Sec}_P(Z) = t-1$, where $t = \text{depth } A_X$.

Next let us analyze this situation in more detail. To this end let $Z = S(a_1, \dots, a_k)$ for certain integers $a_i, i = 1, \dots, k$. Then it follows that

$$r = k - 2 + \sum_{i=1}^k a_i \text{ and } c + 2 = \sum_{i=1}^k a_i,$$

where $c = \text{codim } X$.

As X is not a cone, Z cannot be a cone over a rational normal scroll. Therefore $\min\{a_i : i = 1, \dots, k\} \geq 1$. So, for a given codimension c we have to investigate all the possible partitions

$$c + 2 = \sum_{i=1}^k a_i, \text{ with } k \geq 1 \text{ and } a_1 \geq a_2 \geq \dots \geq a_k \geq 1.$$

For $c = 2$ we thus get the following types of the possible rational normal scrolls:

| k | a_1 | a_2 | a_3 | a_4 | $r + 1$ | $\dim X$ |
|-----|-------|-------|-------|-------|---------|----------|
| 1 | 4 | | | | 4 | 1 |
| 2 | 3 | 1 | | | 5 | 2 |
| 2 | 2 | 2 | | | 5 | 2 |
| 3 | 2 | 1 | 1 | | 6 | 3 |
| 4 | 1 | 1 | 1 | 1 | 7 | 4 |

This proves already that $\dim X \leq 4$.

Next, we discuss the case in which the codimension equals 3. Here there are the following possibilities for the type of the scroll Z :

| k | a_1 | a_2 | a_3 | a_4 | a_5 | $r + 1$ | $\dim X$ |
|-----|-------|-------|-------|-------|-------|---------|----------|
| 1 | 5 | | | | | 5 | 1 |
| 2 | 4 | 1 | | | | 6 | 2 |
| 2 | 3 | 2 | | | | 6 | 2 |
| 3 | 3 | 1 | 1 | | | 7 | 3 |
| 3 | 2 | 2 | 1 | | | 7 | 3 |
| 4 | 2 | 1 | 1 | 1 | | 8 | 4 |
| 5 | 1 | 1 | 1 | 1 | 1 | 9 | 5 |

Therefore $\dim X \leq 5$. We already know that $u_1 \geq 3$. Now, on use of 3.3 we easily get the requested constraints in Theorem 2.2.

Finally, let $\text{codim } X = 4$. Then, as above there is the following list of possible types for the scrolls Z :

| k | a_1 | a_2 | a_3 | a_4 | a_5 | a_6 | $r + 1$ | $\dim X$ |
|-----|-------|-------|-------|-------|-------|-------|---------|----------|
| 1 | 6 | | | | | | 6 | 1 |
| 2 | 5 | 1 | | | | | 7 | 2 |
| 2 | 4 | 2 | | | | | 7 | 2 |
| 2 | 3 | 3 | | | | | 7 | 2 |
| 3 | 4 | 1 | 1 | | | | 8 | 3 |
| 3 | 3 | 2 | 1 | | | | 8 | 3 |
| 3 | 2 | 2 | 2 | | | | 8 | 3 |
| 4 | 3 | 1 | 1 | 1 | | | 9 | 4 |
| 4 | 2 | 2 | 1 | 1 | | | 9 | 4 |
| 5 | 2 | 1 | 1 | 1 | 1 | | 10 | 5 |
| 6 | 1 | 1 | 1 | 1 | 1 | 1 | 11 | 6 |

As above it follows that $\dim X \leq 6$. We know that $u_1 \geq 6$. So, by 3.3 we get the requested constraints in Theorem 2.3.

Finally we prove Corollary 2.4. As $\min\{a_i : 1 \leq i \leq k\} \geq 1$, we have $\text{codim } X + 2 \geq k = \dim X$. This means, X is arithmetically Cohen-Macaulay whenever $\dim X > \text{codim } X + 2$. In view of [2, Section 6] this implies that X is an arithmetically Gorenstein variety. Now according to the classification of Del Pezzo varieties by Fujita (cf. [5, Sections 8 and 9]) it follows that $\dim X \leq 6$. Therefore $\text{codim } X = 3$, so our claim follows by Theorem 2.2.

For the existence of the samples described in Theorems 2.1 and 2.2 we refer to the next section.

We close this section with a result about the number of cubics in a minimal generating set of the defining ideal of a certain variety of almost minimal degree.

Lemma 3.5. *Let $X \subset \mathbb{P}_K^r$ be a variety of almost minimal degree with codepth $A_X = 1$ and $c := \text{codim } X \geq 4$. Then the defining ideal I_X of X is generated by $\binom{c+1}{2} - 2$ quadrics and at most one cubic.*

Proof. First we reduce the problem to the case in which $\dim X = 1$. Let $d = \dim X > 1$. By an argument of Bertini type (cf. [11]) we may find generic linear forms $l_1, \dots, l_{d-1} \in S_1$ such that $W = X \cap \mathbb{P}_K^{c+1} \subset \mathbb{P}_K^{c+1} := \text{Proj}(S/(l_0, \dots, l_{d-1})S)$ is a non-degenerate integral variety of almost minimal degree. As l_0, \dots, l_{d-1} are chosen generically and $\text{depth } A_X = d$ they form an A_X -regular sequence. Therefore the Betti diagrams of I_X and I_W are the same. In particular $\text{codepth } A_W = 1$.

So we assume $X \subset \mathbb{P}_K^s$ with $\dim X = 1$ and $s = c + 1$. The statement about the number of quadrics is a consequence of Lemma 3.2. Since X is of almost minimal degree we know that $I = I_X$ is 3-regular (cf. 3.3). Write $I = (J, LS)$ with $J = I_2 S$ and with a K -vector space $L \subset S_3$ such that $I_3 = J_3 \oplus L$. Our aim is to show that $\dim_K L \leq 1$.

After an appropriate linear coordinate change we may assume that $x_s \in S_1$ is generic. Let $T := S/x_s S = K[x_0, \dots, x_{s-1}]$. Then $R := T/(J, L)T \simeq S/(I, x_s S)$ defines a scheme Z of $s + 1$ points in semi uniform position in \mathbb{P}_K^{s-1} . The short exact sequence $0 \rightarrow A_X(-1) \xrightarrow{x_s} A_X \rightarrow R \rightarrow 0$ induces an isomorphism $H_{R+}^0(R) \simeq K(-2)$. But

this means that the vanishing ideal of Z in T has the form $(J, L, q)T$ with an appropriate quadric $q \in S_2$. Since $s \geq 3$ the minimal free resolution of this ideal has the form

$$T^{a_2}(-3) \xrightarrow{\phi} T^{a_1}(-2) \xrightarrow{\pi} (J, L, q)T \rightarrow 0.$$

This allows us to write $(J, L, q)T = (J, q)T$ and to assume that the first $a_1 - 1$ generators of $T^{a_1}(-2)$ are mapped by π onto a K -basis of $(JT)_2$ and the last generator is mapped by π to $q \cdot 1_T$. Clearly, ϕ is given by a matrix with linear entries. This shows that $M := JT :_T q \subset T$ is a proper ideal generated by linear forms.

As $JT \subseteq M$ and as $(J, qT) = IT$ is of height $s-1$ we must have $s-2 \leq \text{height } M \leq s$. As M is generated by linear forms, $(T/M)_1$ is a K -vector space of dimension $t := s - \text{height } M \in \{0, 1, 2\}$. So the graded short exact sequence

$$0 \rightarrow T/M(-2) \rightarrow T/JT \rightarrow T/(J, q)T \rightarrow 0$$

shows that

$$\dim_K(IT)_3 = \dim_K((J, q)T)_3 = \dim_K(JT)_3 + t.$$

Therefore, we may write $(I, x_s) = (J, L', x_s)$ where $L' \subseteq L \subset S_3$ is a K -vector space of dimension $\leq t$. As I is a prime ideal and as $x_s \in S_1 \setminus I$ it follows $I = (J, L')$, hence $L' = L$. So, if $\dim_K L' \leq 1$, we are done.

Otherwise, $\dim_K L' = \dim_K L = 2 = t$ and we may write $I = (J, k_1, k_2)$ with $k_1, k_2 \in S_3$. As $\text{height } I = s-1$ it follows $\text{height } J \geq s-3$. As $JT \subseteq M$ and as $\text{height } M = s-2$ we have $\text{height } JT \leq s-2$. As x_s is a generic linear form, this means that $\text{height } J \leq s-3$ and hence $\text{height } J = s-3$. As $I = (J, k_1, k_2)$ is a prime ideal of height $s-1 = \text{height } J + 2$, the ideal J must be prime too.

As x_s is generic and $\text{height } J \leq s-3$, we may conclude by Bertini's theorem that $JT \subset T$ defines an integral subscheme of $\mathbb{P}_K^{s-1} = \text{Proj}(T)$. So, the saturation

$$JT :_T \langle T_+ \rangle \subset T \text{ of } JT \text{ in } T$$

is a prime ideal of height $s-2$. As $JT \subseteq M \subset T_+$ and as M is a prime ideal we get $JT :_T \langle T_+ \rangle = M$. Therefore

$$\text{Proj}(T/IT) = \text{Proj}(T/(J, q)T) = \text{Proj}(T/(M, qT))$$

consists of two points, so that $s+1=2$, a contradiction. So, the case $\dim_K L' \geq 2$ does not occur at all. \square

The previous Lemma 3.5 is inspired by a corresponding statement for curves of degree $r+2$ in \mathbb{P}_K^r shown by the authors (cf. [1, Lemma (6.4)]).

Moreover, for the case of codepth $A_X \geq 2$ the number of cubics needed to define X is not bounded by 1 (cf. the examples in [2, Section 9]).

4. EXAMPLES

In this section we want to confirm the existence of the varieties of almost minimal degree $X \subset \mathbb{P}_K^r$ as they were described in the Theorems 2.1, 2.2 and 2.3.

First of all we want to show the existence of a Del Pezzo variety as it required for Theorem 2.3 (a).

Example 4.1. Let $X = \mathbb{P}_K^2 \times \mathbb{P}_K^2 \subset \mathbb{P}_K^8$ be the Segre product of two projective planes. Its defining ideal I_X is generated by the 2×2 -minors of the following generic 3×3 -matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \\ x_6 & x_7 & x_8 \end{pmatrix}.$$

It is easy to see that $\dim X = 4$, $\operatorname{codim} X = 4$ and $\deg X = 6$. Therefore, X is a variety of almost minimal degree. Moreover, A_X is a Cohen-Macaulay and therefore a Gorenstein ring.

In the next examples let us show that the two different Betti diagrams of statement (b) in 2.3 indeed occur. Note that they require $\operatorname{codepth} A_X = 1$ and $\operatorname{codim} X = 4$.

Example 4.2. Consider the rational normal surface scroll $Z = S(3, 3) \subset \mathbb{P}_K^7$. Let $P_1 = (0 : 0 : 0 : 0 : 1 : 0 : 0 : 1)$ and $P_2 = (1 : 0 : 0 : 0 : 0 : 0 : 0 : 1)$ in \mathbb{P}_K^7 . Then $P_i \in \mathbb{P}_K^7 \setminus Z$, $i = 1, 2$, as it is easily seen. Define X_i to be the projection of Z from P_i , $i = 1, 2$. Then $\dim X_i = 2$ and $\operatorname{codepth} A_{X_i} = 1$ for $i = 1, 2$. The Betti diagrams of I_{X_i} , $i = 1, 2$, are those of Theorem 2.3 (b).

Now we construct non-arithmetically Cohen-Macaulay varieties of almost minimal degree of the type mentioned in Theorem 2.2. To this end we use the possible rational normal scrolls $Z = S(a_1, \dots, a_k)$ of the proof of Theorem 2.2 which after appropriate projection furnish the varieties we are looking for. The construction of the examples corresponding to Theorems 2.1 and 2.3 follows similarly, and so we skip the details in these cases.

Example 4.3. Let $Z = S(5) \subset \mathbb{P}_K^5$ denote the rational normal curve of degree 5. Choose $P \in \mathbb{P}_K^5 \setminus Z$ a generic point. Then $X \subset \mathbb{P}_K^4$, the projection of Z from P , is an example of $\dim X = 1$ and $\operatorname{codepth} A_X = 1$.

Next we want to investigate the case of surfaces.

Example 4.4. Let $Z = S(4, 1) \subset \mathbb{P}_K^6$. Consider the two points $P_1 = (0 : 1 : 0 : 0 : 0 : 0 : 0)$ and $P_2 = (0 : 0 : 1 : 0 : 0 : 0 : 0)$. Then $P_i \in \mathbb{P}_K^6 \setminus Z$, for $i = 1, 2$, as easily seen. Let X_i , $i = 1, 2$, denote the projection of Z from P_i . Then $\operatorname{codepth} A_{X_1} = 1$ and $\operatorname{codepth} A_{X_2} = 2$. The same type of examples may be produced by projections of the scroll $S(3, 2)$.

Example 4.5. Let $Z = S(3, 1, 1) \subset \mathbb{P}_K^7$. Consider the points $P_1 = (0 : 1 : 0 : 0 : 0 : 0 : 0 : 0)$ and $P_2 = (0 : 0 : 0 : 0 : 1 : 1 : 0 : 0)$. Then it is easy to see that $P_i \in \mathbb{P}_K^7 \setminus Z$, $i = 1, 2$. Let X_i denote the projection of Z from P_i . Then $\operatorname{codepth} A_{X_1} = 2$ and $\operatorname{codepth} A_{X_2} = 1$. The same type of examples may be produced by projections from the scroll $S(2, 2, 1)$. This covers the examples of $\dim X = 3$.

Now let us consider the situation of fourfolds.

Example 4.6. Consider the scroll $Z = S(2, 1, 1, 1) \subset \mathbb{P}_K^8$. Let $P_1 = (0 : 1 : 0 : 0 : 0 : 0 : 0 : 0 : 0)$ and $P_2 = (0 : 0 : 0 : 0 : 0 : 0 : 1 : 1 : 0)$. Then $P_i \in \mathbb{P}_K^8 \setminus Z$, $i = 1, 2$. Let $X_i \subset \mathbb{P}^7$ denote the projection of Z from P_i , $i = 1, 2$. Then $\operatorname{codepth} A_{X_1} = 2$, while $\operatorname{codepth} A_{X_2} = 1$.

Finally let us consider the case of $\dim X = 5$.

Example 4.7. Let $Z = S(1, 1, 1, 1, 1) \subset \mathbb{P}_K^9$ be the Segre variety. Then $P \in \mathbb{P}_K^9 \setminus Z$ for the point $P = (0 : 1 : 1 : 0 : 0 : 0 : 0 : 0 : 0 : 0)$. Let $X \subset \mathbb{P}_K^8$ denote the projection of Z from P . Then $\text{depth } A_X = 4$, and therefore $\text{codepth } A_X = 2$. Finally we have to observe that $\text{codepth } A_X = 1$ is impossible if $\dim X = 5$, as $\text{depth } A_X \leq 4$ (cf. Lemma 3.1).

The Examples 4.3 – 4.7 provide the existence of the samples claimed in Theorem 2.2. Similar constructions provide varieties as mentioned in Theorem 2.1 and 2.3.

In the final examples, we will show that in higher codimension the shape of the Betti diagram of I_X for a variety of minimal degree may vary in a much stronger way. In fact the "beginning of the Betti diagrams" may be rather different from each other.

Example 4.8. Let $Z = S(8) \subset \mathbb{P}_K^8$ denote the rational normal curve of degree 8. Let $P_1 = (0 : 0 : 0 : 0 : 0 : 0 : 1 : 0 : 0)$, $P_2 = (0 : 0 : 0 : 0 : 0 : 1 : 0 : 0 : 0)$ and $P_3 = (0 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 0)$. Then $P_i \in \mathbb{P}_K^8 \setminus Z$ for $i = 1, 2, 3$. Let $X_i \subset \mathbb{P}_K^7$, $i = 1, 2, 3$, denote the projection of Z from P_i . Then the Betti diagrams of I_{X_i} , $i = 1, 2, 3$, resp. have the form:

| i | | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|----|----|----|----|----|---|---|
| 1 | 1 | 19 | 58 | 75 | 44 | 5 | 0 | 0 |
| | 2 | 1 | 6 | 15 | 20 | 21 | 8 | 1 |
| 2 | 1 | 19 | 57 | 70 | 34 | 5 | 0 | 0 |
| | 2 | 0 | 1 | 5 | 20 | 21 | 8 | 1 |
| 3 | 1 | 19 | 57 | 69 | 34 | 5 | 0 | 0 |
| | 2 | 0 | 0 | 5 | 20 | 21 | 8 | 1 |

In all three cases $\text{codim } X_i = 4$ and $\text{codepth } A_{X_i} = 1$, $i = 1, 2, 3$. Remember that the number of cubics in the defining ideals is bounded by 1 (cf. 3.5). It follows that X_1 is contained in the scroll $S(5, 1)$, while X_2 is contained in the scroll $S(4, 2)$ and X_3 is contained in the scroll $S(3, 3)$.

In view of the Example 4.8 and corresponding examples in higher dimensions one might expect that the type of the rational normal scroll that contains the variety of almost minimal degree determines the Betti diagram "near the beginning of the resolution". In small codimensions, the different types of such scrolls are much more limited than imposed by Theorems 2.1, 2.2 and 2.3. It is a big challenge to understand the rôle of the scrolls for the beginning of the minimal free resolution.

Moreover the examples in 4.8 show that the estimates for the Betti numbers at the beginning of the Betti diagram (cf. Lemma 3.3) are fairly weak.

Remark 4.9. To compute the Betti diagrams and hence the arithmetic depths of the above examples, we have made use of the computer algebra system SINGULAR (cf. [8]). Moreover, there is in preparation a conceptual approach for the computation of the $\text{depth } A_X$ in terms of the center of the projection and the secant variety of $S(a_1, \dots, a_k) \subset \mathbb{P}_K^{r+1}$.

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